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ROLAND HÄGGKVIST and ANDREW THOMASON

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Let $f(n)$ be the smallest integer such that every tournament of order $f(n)$ contains every oriented tree of order n. Sumner has just conjectures that $f(n) = 2n - 2$, and F. K. Chung has shown that $f(n) \leq (1 + o(1))n \log_2 n$. Here we show that $f(n) \leq 12n$ and $f(n) \leq (4 + o(1))n$.

1. Introduction

It is clear that every sufficiently large tournament contains all oriented trees of order n. Indeed, Fan Chung noticed that every tournament of order $(1+o(1))n \log_2 n$ has this property, since a tournament of order n has a dominating set of order $\lceil \log_2 n \rceil$. Sumner [see 1] has conjectured that every tournament of order $2n - 2$ has this property, and this has been proved by Reid and Wormald [1] in the special case where the tree is a caterpillar with a directed spine.

Our aim in this paper is to find an absolute constant c such that all tournaments of order cn contain all oriented trees of order n. In the first theorem we show that $c \leq 12$, in the third we improve this to $c \leq 4 + o(1)$. Furthermore we give sharper estimates on the orders of tournaments containing all oriented trees of order n with at most k end-vertices.

2. Notation

If G is an oriented graph, or for that matter a digraph, an edge from a vertex x in $V(G)$ to a vertex y in $V(G)$ is denoted by xy, the notation $N_G^+(x)$ for x in $V(G)$ stands for the set $\{y : xy \in E(G)\}$ and $E_G^+(x)$ denotes the set $\{xy : xy \in E(G)\}$. We put $d_G^+(x) = |N_G^+(x)| = |E_G^+(x)|$ and define N^-, E^- and d^- similarly, dropping the subscript G most of the time. The induced oriented subgraph of G spanned by a set $X \subset V(G)$ is denoted $G[X]$. For an oriented tree T and a vertex $v \in V(T)$ we define $T^+(v)$ to be the set of components of $T - v$ incident with $N^+(v)$ and similarly we define $T^-(v)$.

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3. Main theorems and proofs

The central idea of our proofs is that of the *p-heart* $h_p(T)$ of an n-order tree T, defined (for an integer $p \geq 2$) to be the tree spanned by those edges e in $E(T)$ for which each of the two components of $T-e$ has order at least n/p. If there are no such edges, $h_p(T)$ will denote the unique vertex of T whose deletion leaves components each of order less than n/p. Note that $h_p(T)$ is connected, so it is indeed a tree. Moreover, for $p \geq 3$, $h_p(T)$ has at most $p-1$ endvertices, because the removal of an end-edge of $h_p(T)$ from T leaves a component of T order at least n/p, and so if $h_p(T)$ has k endvertices then $h_p(T)$ has order at most $n - k\lceil n/p \rceil + k$. Note furthermore that $T - h_p(T)$ has all components of order less than n/p, that is, at most $\lceil n/p \rceil - 1$.

Our approach to finding an oriented tree T is some large tournaments S is first to find the p-heart $h_p(T)$ of T in a subtournament of S spanned by vertices of large indegree and outdegree. We then find the components of $T - h_p(T)$ in the remainder of S, in such a way that these components are joined in S to the end-vertices of $h_p(T)$ so as to form T. The existence of the p-heart of T in S is established by using results concerning trees with a bounded number k of endvertices. We begin by proving a simple lemma which we state in a general form, although we shall use only the cases $k \leq 5$. A much stronger form of the lemma (at least for n large compared with k) will be proved as Theorem 8 below, but for now let us be content with the following.

Lemma 1. *Every oriented tree of order n with k end-vertices is contained in every tournament of order $2^{k-2}n+1$.*

Proof. For $k=2$, the lemma states that every oriented path of order n is contained in every tournament of order $n+1$, which is precisely Corollary 2 in [2]. So we may proceed by induction on k. Suppose that T is a tree of order n with $k \geq 3$ end-vertices, and let S be a tournament of order $2^{k-2}n+1$. Choose a branch-vertex v of T (what is, a vertex of total degree at least three), and note that each component of $T-v$ has at most $k-1$ end-vertices. Put $a^+ = 2^{k-3}|T^+(v)|+1$ and $a^- = 2^{k-3}|T^-(v)|+1$. Since at most $2a^+ - 1$ vertices of the tournament have outdegree less than a^+, and at most $2a^- - 1$ have indegree less than a^-, there are at least

$$2^{k-2}n + 1 - 2^{k-2}(|T^+(v)| + |T^-(v)|) - 2 = 2^{k-2}n - 2^{k-2}n + 2^{k-2} - 1 \geq 1$$

vertices $u \in V(S)$ with $d_S^+(u) \geq a^+$ and $d_S^-(u) \geq a^-$. Choose one such vertex. By the induction hypothesis we may find $T^+(v)$ in $S[N_S^+(u)]$ (finding the components one by one if necessary) and $T^-(v)$ in $S[N_S^-(u)]$. Hence we find T in S as required. (In fact we can find T in any tournament of order $2^{k-2}n - (4^{k-2}-1)/3 + 2^{k-2}$, by the same method, but we shall not need this.) ∎

We are now ready to prove our first Theorem (the Theorem will be improved for large n by Theorem 10).

Theorem 2. *Every oriented tree of order n is contained in every tournament of order $12n$.*

Proof. Suppose that the theorem is false, and let n be the smallest integer for which it fails. Clearly $n \geq 5$. Let T be an oriented tree of order n and S a tournament of order $12n$ which fails to contain a copy of T. Let t be the order of $H = h_6(T)$. Recall that H has $k \leq 5$ endvertices and that $t \leq n - k\lceil n/6 \rceil + k$. By examining the four cases $2 \leq k \leq 5$, plus the case $t = 1$, we see by Lemma 1 that H can be found in any tournament of order at least $9n - t - 44\lceil n/6 \rceil + 50$. For instance, in the case $k = 4$, the most critical case, we know that H can be found in any tournament of order $4t + 1$. We must then check the inequality

$$4(n - 4\lceil n/6 \rceil + 4) + 1 \leq 9n - (n - 4\lceil n/6 \rceil + 4) - 44\lceil n/6 \rceil + 50$$

or

$$0 \leq 4n - 24\lceil n/6 \rceil + 29,$$

which is correct.

For each vertex $v \in V(H)$ let $U^+(v)$ be the set of components of $T - H$ incident with $N_T^+(v)$ and define $U^-(v)$ similarly. We may suppose that $\sum_v |U^-(v)| \leq \sum_v |U^+(v)|$, so that $\sum_v |U^-(v)| \leq \lfloor (n-t)/2 \rfloor$.

Now set $a^- = 11\lceil n/6 \rceil + \lfloor (n+t)/2 \rfloor - 12$ and $a^+ = 11\lceil n/6 \rceil + n - 12$. There are in S at least

$$12n - 2a^- + 1 - 2a^+ + 1 \geq 9n - t - 44\lceil n/6 \rceil + 50$$

vertices w with $d_S^+(w) \geq a^+$ and $d_S^-(w) \geq a^-$. Choose some copy H^* of H spanned by these vertices. We finish by finding copies of $U^-(v)$ and $U^+(v)$ is S joined appropriately to H^*. This works if we first find all $U^-(v)$'s for $v \in V(H)$, and then look for the U^+'s. To see this, note that when we seek $U \in U^-(v)$ joined to the vertex $w \in V(H^*)$ corresponding to the vertex $v \in V(H)$ there are at least

$$d^-(w) - t + 1 - \left(\sum_v |U^-(v)| - |U| \right) \geq 11(\lceil n/6 \rceil - 1) + |U| \geq 12|U|$$

vertices of $N_S^-(w)$ so far unused. Then when we return to insert some $U' \in U^+(v)$ we see that

$$d^+(w) - t + 1 - \left(\sum_v |U^-(v)| + |U^+(v)| - |U'| \right) \geq 11(\lceil n/6 \rceil - 1) + |U'| \geq 12|U'|$$

and so likewise U' may be found.

We thereby find a copy of T in S, contradicting the choice of T as a minimal counterexample to the theorem, whence no such counterexample can exist, and the theorem is proved. ∎

The choice of $p = 6$ in the proof above was not critical; any value of $p \geq 5$ will yield some constant $c = c(p)$ such that all tournaments of order cn contain all oriented trees of order n. In fact the choice $p = 7$ yields $c = 35/4$, which is better than that claimed in Theorem 2. However the use of larger values of p results in a deterioration, because of the need to use the feeble Lemma 1 for large k.

In order to improve Theorem 2 for large n, it will be enough to improve Lemma 1 in the case when n is large compared with k. We shall aim to find a function $g(k)$, such that the function $2^{k-2}n + 1$ in the lemma can be replaced by $n + g(k)$. Now in an oriented tree T of (large) order n, with a small number k of endvertices, most vertices are of (total) degree 2. The tree can then be thought of as a collection of at most $2k-3$ oriented paths fastened together on some way. Provided these paths are not directed they can easily be found in our tournament by means of the following Proposition, which is a weak amalgam of Theorems 3, 4, and 5 of [2].

Proposition 3. *Let P be an oriented path of order n with at least two blocks (maximal directed subpaths), whose first and last blocks have lengths k and l respectively. Let S be a tournament of order $n+5$, and let K, L be disjoint subsets of $V(S)$ with $|K| = k+3$ and $|L| = l+3$. Then there is a copy of P in S, with its initial vertex in K and its end vertex in L.*

Different argument are needed to find the paths of T if they are directed, that is, consist of a single block, and this is where the excess $g(k)$ vertices will be needed. But it is time now to turn technical. The reader may prefer to skip Lemma 6, or even to Theorem 8, on a first reading.

Given a tournament S, and disjoint subsets $X, Y \subset V(S)$, the notation $X \Rightarrow Y$ means $xy \in E(S)$ for all $x \in X$ and $y \in Y$. The notation may be abused somewhat; for instance if $X = \{x\}$ we may write $x \Rightarrow Y$, or if $R = S[X]$ we may write $R \Rightarrow Y$. The notation $R = \langle x_1, \ldots, x_r \rangle$ means R is a transitive tournament with $V(R) = \{x_1, \ldots, x_r\}$ and $x_i x_j \in E(R)$, $1 \le i < j \le r$.

Lemma 4. *Given a tournament S and an integer p, $0 \le p \le |S|-1$, there exists a directed path $P = x_1 x_2 \ldots x_p$ (empty if $p = 0$) in S and a maximum transitive subtournament $R = \langle x_1, \ldots \rangle$ in $S - P$ such that $x_p x \in E(S)$.*

Proof. The proof is by induction on S; the lemma is trivial if $|S| = 1$. If $|S| > 1$ we choose a maximum transitive subtournament $R' = \langle v, \ldots \rangle$ of S. If $|N_S^-(v)| \ge p$ we are home by choosing a directed path $x_1 x_2 \ldots x_p$ in $S[N_S^-(v)]$ and letting $R = R'$, $x = v$. If $|N_S^-(v)| < p$ we choose a Hamilton path $x_1 x_2 \ldots x_{q-1}$ in $S[N_S^-(v)]$, set $x_q = v$, and by induction find $x_{q+1} x_{q+2} \ldots x_p$ and R in $S - N_S^-(v) - v$. ∎

Let $m(S)$ be defined as the maximum order of a transitive subtournament of S. It is easily shown that if $|S| \ge 2^m$ then $m(S) \ge m+1$.

Lemma 5. *Let $u \ge 1$, $m \ge 0$ and $p \ge 3$ be integers, let S be a tournament of order at least $u + p + 2^{m+u+2}$, and let R be a transitive subtournament of S with $|R| \ge m(S) - m$. Then there exists a set $U \subset V(R)$, $|U| = u$, a directed path $P = v_1 v_2 \ldots v_p$ is $S - U$, a tournament $S' \subset S - U - P$ and a transitive subtournament R' in S', such that $U \Rightarrow v_1$, $v_p \Rightarrow R'$, $U \Rightarrow R'$, $|S'| \ge |S| - u - p - 2^{m+u+2}$ and $|R'| \ge m(S') - m'$, where $m' = m + u + 3$.*

Proof. Let R be $\langle x_1, x_2, \ldots, x_r \rangle$. We note that $|S| \ge 2^{m+u+2}$ whence $m(S) \ge m + u + 3$ and $r \ge u + 3$. Let $Z = \{x_1, \ldots, x_{u+2}\}$, let

$$Y = \{v \in V(S-R) \; : \; vx_i \in E(S) \text{ for all } x_i \in R - Z\}$$

and

$$X = \{v \in V(S-R-Y) \; : \; vx_i \in E(S) \text{ for some } x_i \in Z\}$$

Note that $|Y| < 2^{m+u+2}$, for otherwise $S[Y]$ contains a transitive subtournament of order $m+u+3$, which together with $R - Z$ forms a transitive subtournament of order $m(S)+1$, a contradiction. We proceed in one of two ways, depending on the order of X.

If $|X| \geq p-2$, we choose a directed path $v_2v_3\dots v_{p-1}$ in $S[X]$. Since $X \cap Y = \emptyset$, there is an $x_i \in R - Z$ such that $x_iv_2 \in E(S)$. Put $v_1 = x_i$. Likewise there is an $x_j \in Z$ with $x_{p-1}x_j \in E(S)$, since $x_{p-1} \in X$, and we put $v_p = x_j$. Now choose $U \subset Z - x_j$ with $|U| = u$, and let R' and S' be the tournaments $(R-Z)-x_i$ and $S - U - P - Y$ respectively. Then clearly

$$U \Rightarrow v_1, \quad U \Rightarrow R', \quad v_p \Rightarrow R', \quad |R'| \geq |R| - u - 3 \geq m(S) - m' \geq m(S') - m'$$

and $|S'| \geq |S| - u - p - 2^{m+u+2}$.

Otherwise, $|X| = q \leq p-3$. If $q = 0$ let $U = \{x_1, x_2, \dots, x_u\}$ and $v_1v_2 = x_{u+1}x_{u+2}$; if not, choose a directed path $v_2v_3\dots v_{q+1}$ in $S[X]$. In the latter case there are as in the preceding paragraph vertices $x_i \in R - Z$ and $x_j \in X$ with $x_iv_2 \in E(S)$ and $v_{q+1}x_j \in E(S)$. We put $v_1 = x_i$ and $v_{q+2} = x_j$, and choose $U \subset Z - x_j$ with $|U| = u$. Hence in both cases we find U such that $|U| = u$, a path $P' = v_1v_2\dots v_{q+2}$ containing X with $U \Rightarrow v_1$, and such that $(U \cup \{v_{q+2}\}) \Rightarrow S - Z - P' - Y$ (by the definition of X). Applying Lemma 4 to $S - Z - P' - Y$ we find a directed path $P^* = v_{q+3}v_{q+4}\dots v_{p-1}$ and a maximum transitive subtournament $R^* = \langle v_p, \dots\rangle$ of $S - Z - P' - Y - P^*$, with $v_{p-1}v_p \in E(S)$. Note that $v_{q+2}v_{q+3} \in E(S)$ since $v_{q+2} \Rightarrow S - Z - P' - Y$, so we may put $P = v_1v_2\dots v_p$ and $R' = R^* - v_p$. Then $v_p \Rightarrow R'$, $U \Rightarrow R'$, and putting $S' = S - Z - P - Y$ we get $|S'| > |S| - (u+p+1) - 2^{m+u+2}$ and $|R'| \geq m(S') - m'$. ∎

Lemma 6. *If T is an oriented tree of order n we may label the vertices of T as x_1, x_2, $\dots$, x_n and the edges as e_1, e_2, $\dots$, e_{n-1} such that if $e_i = x_jx_{j'}$, then $j \leq i < j'$.*

Proof. Induction on $|T|$. Choose an end-vertex x of T. If yx is an edge of T we label the vertices of $T - x$ as x_1, x_2, $\dots$, x_{n-1} in the approved way, and put $x_n = x$, $e_{n-1} = yx$. Otherwise, if xy is an edge of T we label the vertices of $T - x$ as x_2, x_3, $\dots$, x_n and the edges as e_2, e_3, $\dots$, e_{n-1} such that the lemma holds, and we put $x_1 = x$, $e_1 = xy$. ∎

Lemma 7. *Suppose that the oriented tree T of order n has k end-vertices, and furthermore that the oriented paths in T induced by the vertices of degree 2 in the underlying tree of T are directed; that is, for each $v \in V(T)$, either $d^+(v) = d^-(v) = 1$ or else v is an end-vertex or a branch-vertex or adjacent to an end-vertex or branch-vertex. Then T is contained in any tournament of order $n + 2^{8k^3}$.*

Proof. First note that the tree T has at most $k-2$ branch-vertices. Let M be the subforest of T induced by the end-vertices, branch-vertices, and all vertices of distance at most $k+1$ from an end-vertex or branch-vertex. Then M has $c \leq 2k-2$ components, and since each component is the union of at most $2k-3$ paths of length at most $2k+3$, each component has order at most $(2k-3)(2k+3)+1 \leq 4k^2$. The edges of T not in M consist of $c-1$ directed paths between the components of M. Applying Lemma 6 to the tree T^* whose vertices are the components of M, with an edge in T^* from the vertex C to the vertex C' if and only if there is a path from C to C' in T, we see that we may label the components of M as C_1, C_2, $\dots$, C_c, and the paths of $T - M$ by P_1, P_2, $\dots$, P_{c-1} such that if P_i joins C_j to $C_{j'}$ then $j \leq i < j'$.

Now the endvertices of P_i are of distance at least $k+1$ from any branch-vertex or end-vertex of T. So $C_j \cup P_i \cup C_{j'}$ contains a directed path

$$u_j^i \dots u_i^i v_1^i v_2^i \dots v_{p_i}^i u_{i+1}^i \dots u_{j'}^i,$$

each vertex having degree two in T, with $u_j^i \in C$, $u_{j'}^i \in C_{j'}$, and $p_i \geq 3$. Let

$$C_j^* = C_j \setminus \{u_l^i;\ 1 \leq i \leq c-1,\ 1 \leq l \leq c\} \qquad \text{for} \qquad 1 \leq j \leq c.$$

If follows that a tournament will contain T if it contains c transitive tournaments U_1, $\dots$, U_c with $|U_i| = 4k^2 + c$, and $c-1$ directed paths $P_i^* = v_1^i v_2^i \dots v_{p_i}^i$, all disjoint, with $U_i \Rightarrow v_1^i$, $v_{p_i}^i \Rightarrow U_{i+1}$ and $U_i \Rightarrow U_{i+1}$, $1 \leq i \leq c-1$. For then C_j^*, along with any required vertices u_j^i, $1 \leq i \leq c-1$, can be found in U_j.

We complete the proof by showing any tournament S of order $n + 2^{8k^3}$ contains transitive subtournaments $U_1, \dots, U_c$, and paths $P_i^* = v_1^i v_2^i \dots v_{p_i}^i$ as described. Let $u = 4k^2 + c$ and $m_i = (i-1)(u+3)$, $1 \leq i \leq c$. By $c-1$ applications of Lemma 5 we find tournaments $S = S_1, S_2, \dots, S_c$ with $|S_{i+1}| \geq |S_i| - u - p_i - 2^{m_i+u+2}$, transitive subtournaments $R_i \subset S_i$ with $|R_i| \geq m(S_i) - m_i$, sets $U_i \subset V(R_i)$ with $|U_i| = u$, and paths $P_i^* = v_1^i v_2^i \dots v_{p_i}^i$, such that $S_{i+1} \subset S_i - U_i - P_i^*$ and the U_j and P_i^* satisfy the conditions of the previous paragraph. So it will be enough to show that $|S_i| \geq u + p_i + 2^{m_i+u+2}$, so that the lemma can be applied for the i-th time $(i \leq c-1)$, and that $|S_c| \geq 2^{m_c+u}$, so that $|R_c| \geq u$ enabling us to find U_c.

However,

$$|S_i| \geq |S| - (i-1)u - \sum_{j<i} p_j - \sum_{j<i} 2^{m_j+u+2},$$

so the inequality $|S_i| \geq u + p_i + 2^{m_j+u+2}$ follows if

$$|S| - iu - \sum_{j \leq i} p_j - \sum_{j \leq c} 2^{m_j+u+2} \geq 0 .$$

Hence it is sufficient to verify the second of these. But $\sum_{j<c} p_j < n$ so the left hand side is at least

$$\begin{aligned} 2^{8k^3} - (c-1)u - 2^{u+2}\sum_{j \leq c} 2^{m_j} &\geq 2^{8k^3} - 2^{(c-1)u} - 2^{u+2}\left(\frac{2^{c(u+3)}-1}{2^{u+3}-1}\right) \\ &\geq 2^{8k^3} - 2^{(c-1)u} - 2^{c(u+3)} \\ &\geq 2^{8k^3} - 2^{c(u+3)+1} \\ &\geq 0 \end{aligned}$$

since $c \leq 2k-2$ and $u = 4k^2 + c$. ∎

We are now ready to prove a theorem which shows that the Sumner conjecture is far from sharp for trees with few end-vertices.

Theorem 8. *Let T be an oriented tree of order n with k endvertices. Then T is contained in every tournament of order $n + 2^{512k^3}$.*

Proof. Consider a maximal oriented path P in T, each of whose vertices, including the end-vertices, have degree 2 in the underlying tree of T. If P is not directed, we let P^* be a maximal subpath of P having distinct end-blocks each of length 1. Let the end-vertices of P^* be u and v, with u adjacent to $x \in V(T - P^*)$ and v adjacent to $y \in V(T - P^*)$ in the underlying undirected tree of T. We then remove P^* from T and add to $T - P^*$ new vertices u_j and v_j, $1 \le j \le 4$ plus edges u_jx (or xu_j) and yv_j (or v_jy), $1 \le j \le 4$, according as $ux \in E(T)$ (or $xu \in E(T)$) and $yv \in E(T)$ (or $vy \in E(T)$). If we do the appropriate operation for all such non-directed paths P the components of the resulting forest each have at most $4k$ end-vertices and satisfy the condition of Lemma 7 with k replaced by $4k$. Thus we may find this forest in any tournament of order $n + 2^{8(4k)^3} = n + 2^{512k^3}$. But now by Proposition 3 above we may find the paths P^* with end-vertices in the sets $\{u_1, u_2, u_3, u_4\}$ and $\{v_1, v_2, v_3, v_4\}$, and so we find T.

Lemma 9. *Let b, c and t be positive integers, and let S be a tournament of order $2b + 2c + 4t - 5$. Then $V(S)$ contains a set X, $|X| = t$ such that $d^+_{S-X}(v) \ge b$ and $d^-_{S-X}(v) \ge c$ for each vertex $v \in X$.*

Proof. We choose the elements $x_1, x_2, \dots, x_t$ as follows. Having chosen $x_1, x_2, \dots, x_{i-1}$, let $S' = V(S) - \{x_1, x_2, \dots, x_{i-1}\}$. Then, since $|S'| \ge 2(b+t-i)+2(c+t-i)-1$ we may find $x_i \in S'$ with $d^+_{S'}(x_i) \ge b + t - i$ and $d^-_{S'}(x_i) \ge c + t - i$. Clearly X has the desired properties. ∎

Theorem 10. *Let T be an oriented tree of order n. Then T is contained in any tournament S of order at least $4n(1 + 11/k + 2^{512k^3}/n)$ for each $k \ge 3$.*

Proof. Let T^* be the k-heart of T and put $t = |T^*| + 2^{512k^3}$. Let $b = c = n - |T^*| + 11\lfloor n/k \rfloor$. By Lemma 9 we may find a subset X in S, $|X| = t$ with $d^+_{S-X}(v) \ge b$ and $d^-_{S-X}(v) \ge c$ for each $v \in X$. Be Theorem 8 we find T^* in X. But then, as in the proof of Theorem 2, the conditions on X enable us to find the rest of T in $S - X$, since each component of $T - T^*$ has order at most $\lfloor n/k \rfloor$ and can be found in any tournament of order $12\lfloor n/k \rfloor$ (by Theorem 2). ∎

Corollary. $f(n) \le 4n(1 + o(1))$.

Proof. By choosing $512k^3 \approx \log_2 n - \log_2 \log n$ in Theorem 10 we have

$$f(n) \le 4n\left(1 + \frac{c}{(\log n)^{1/3}} + \frac{1}{\log n}\right)$$

for some constant c. ∎

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Roland Häggkvist

University of Stockholm

rolandh@se.umu.umdc.biovax

Andrew Thomason

University of Cambridge

agt2@uk.ac.cam.phx